# DEFINABLE ADDITIVE SUBGROUPS OF PARTIAL DIFFERENTIAL FIELDS OF CHARACTERISTIC ZERO WITH AN AUTOMORPHISM

RONALD F. BUSTAMANTE MEDINA

ABSTRACT. The theory of partial differential fields of characteristic zero with an automorphism has a model companion denoted $\mathrm{DCF}_m\mathrm{A}$. This theory is not complete, but its completions are easily discribed. They are supersimple and eliminate imaginaries. In this paper we prove that the definable subgroups if the additive group of a model of $\mathrm{DCF}_m\mathrm{A}$ are not 1-based and we study the prolongations and arc spaces of the diagonal subgroups.

## 1. INTRODUCTION

Let $(K, \Delta)$ be a differential field of characteristic zero where $\Delta = \{D_1, ..., D_n\}$. We consider the theory of differential fields with $n$ derivations $DF_n$ on the language $\mathcal{L}_n = \mathcal{L} \cup \Delta$, where $\mathcal{L}$ is the language of rings.

In [4], the author proved that $DF_n$ has a model companion, the theory of differentially closed fields $DCF_n$, and showed that it is a complete $\omega$-stable theory which eliminates quantifiers and imaginaries. It has a Noetherian topology, known as $\Delta$-topology or Kolchin topology, which is defined by zeros of ideals of differential polynomials

Now we consider an automorphism $\sigma$ of $K$ that commutes with all $D_i$. In [3] the author showed that in the language $\mathcal{L}_n \cup \{\sigma\}$ , the class of such fields has a model companion. We denote it by $D_nCFA$.

**Fact 1.1.** *(2.1 of* [3]*) Let $(K, \Delta, \sigma)$ be a differential-difference field. Then $(K, \Delta, \sigma)$ is existentially closed if and only if*

1. *$(K, \Delta)$ is a model of $DCF_n$*
2. *Suppose $V$ and $W$ are irreducible $\Delta$-closed sets such that $W \subseteq V \times V^\sigma$ and $W$ projects $\Delta$-dominantly onto both $V$ and $V^\sigma$. If $O_V$ and $O_W$ are nonempty $\Delta$-open sets of $V$ and $W$, respectively, then there is $a \in O_V$ such that $(a, \sigma a) \in O_W$.*

It is not known if this is a first order axiomatisation, but 2.3 of [3] the author provides a first-order axiomatisation.

The model theory of $D_nCFA$ is similar to the model theory of *DCFA*: it is not complete but its completions are easily described, its models eliminate imaginaries, are not stable but are supersimple and quantifier-free stable. Section 2 of [1] can

*Date*: October, 2020.

2000 *Mathematics Subject Classification.* 11U09, 12H05, 12H10 .

*Key words and phrases.* Model theory of fields, supersimple theories, difference-differential fields, definable groups, abelian groups.

be easily generalised to prove that the $SU$-rank of a generic of a model of $D_nCFA$ is $\omega^{n+1}$.

Let $\mathcal{C}_i = \{x \in K : D_i x = 0\}$ be the field of constants of $D_i$ . The field of total constants is $\mathcal{C} = \cap \mathcal{C}_i$. Let $\mathrm{Fix}\sigma = \{x \in K : \sigma(x) = x\}$ be the fixed field.

In [2] we proved Zilber's dichotomy for $\mathrm{DCF}_m\mathrm{A}$:

**Fact 1.2.** *Let l $(\mathcal{U}, \Delta = (D_1, \cdots, D_n), \sigma)$ be a saturated model of $D_nCFA$ and let $p$ be a type over $K \subseteq \mathcal{U}$ of $SU$-rank 1. Then $p$ is either 1-based or non-orthogonal to* $\mathrm{Fix}\sigma \cap \mathcal{C}$.

## 2. Definable Subgroups of the Additive Group

We work on a saturated model $(\mathcal{U}, \Delta = (D_1, \cdots, D_n), \sigma)$ of $D_nCFA$. Recall that the additive group $\mathbb{G}_a^m$ is $\mathcal{U}^m$ with the field addition.

As in [2], we can reduce some questions on definable groups to questions on quantifier-free definable groups. We repeat the contruction for sake of clarity.

Let $a \in \mathcal{U}$ and let $K = acl(K)$ a subfield of $\mathcal{U}$. The dimension of $a$ over $K$ is $tr.dg(K(a)_{\Delta,\sigma}/K)$ where $K(a)_{\Delta,\sigma}$ is the partial differential-difference field generated by $K$ and $a$.

Let $A$ be a definble set in $\mathcal{U}$ defined over $K$, the dimension of $A$ is the dimension of a generic point of $A$ over $K$.

Let $G$ be a connected $\Delta$-group defined over $K = acl(K)$, let $G^{(n)} = G \times G^{\sigma} \times \cdots \times G^{\sigma^n}$ and let $q_n : G \to G^{(n)}$ such that $q(x) = (x, \sigma(x), \cdots, \sigma^n(x)$.

Let $g$ be a generic of $G$ such that $q_n(g)$ is $\Delta$-independent over $K$. Then $q_n(g)$ is a generic of $G^{(n)}$ an thus $q_n(G)$ is $\Delta$-dense in $G^{(n)}$. This implies that $G^{(n)}$ is connected.

Let $H$ be a definable subgroup of $G$ and let $H^{(n)}$ be the $\Delta$-Zariski closure of $q_n(H)$ in $G^{(n)}$. Let $\tilde{H}^{(n)} = \{x \in G : q_n(x) \in H^{(n)}\}$. These groups form a decreasing sequence of quantifier-free definable subgroups of $G$ containing $H$. By Noetherianity this sequence is finite, so there is $N$ such that $\tilde{H}^{(N)} = \cap_n \tilde{H}^{(n)}$. Thus this is the Zariski closure of $H$ in $G$.

**Lemma 2.1.** *Let $G$ be a connected $\Delta$-definable group. Let $H$ be a $\Delta$-definable subgroup of $G$ defined over a field $K = acl(K)$ and let $\bar{H}$ its $(\Delta, \sigma)$-Zariski closure. Then $[\bar{H} : H] < \infty$.*

*Proof.* For $g, h \in G$, $g \mathop{\smile\!\!\!\!\!\!\,|}_K h$ if and only if for all $n$, $q_n(g) \mathop{\smile\!\!\!\!\!\!\,|}_K q_n(h)$ (in $D_nCF$). Then $g$ is a generic of $H$ if and only if $q_n(g)$ is a generic of $H^{(n)}$ (in $D_nCF$). Then $g$ is a generic of $\bar{H}$.

This implies that $SU(H) = SU(\bar{H})$ ant thus $[\bar{H} : H]$ is finite. □

Recall that the additive group $\mathbb{G}_a^m$ is $\mathcal{U}^m$ with the field addition.

As every algebraic subgroup of a vector group is defined by linear equations, then every differential subgroup of a vector group is defined by linear differential equations. Let $H$ be a definable subgroup of $\mathbb{G}_a^m$. Then, each $\tilde{H}^n$ is defined by linear differential equations, and this implies that $H$ is defined by linear $(\sigma, \Delta)$-equations. This implies that $H$ is stable by multiplication by $\mathrm{Fix}\sigma \cap \mathcal{C}$, hence it is a $(\mathrm{Fix}\sigma \cap \mathcal{C})$-vector space.

We have then:

**Lemma 2.2.**

*Every definable subgroup of $\mathbb{G}_a^n$ is quantifier-free definable.*

**Theorem 2.3.** *Let $H$ be a definable subgroup of $\mathbb{G}_a^n$. Then $H$ is not 1-based.*

*Proof.* We have shown that $H$ is quantifier-free definable and is a Fix$\sigma \cap \mathcal{C}$-vector space. Let $H'$ be a definable subgroup of $H$. The set of (linear) $(\sigma, \Delta)$-equations that define $H'$ define an homomorphism from $H$ to a power of $\mathbb{G}_a$ with kernel $H'$. This implies that $H$ has a definable subgroup which is definably isomorphic to $Fix\sigma \cap \mathcal{C}$. By the dichotomy $H$ is not 1-based. □

## 3. Diagonal Subgroups

Let $H = \{x \in \mathcal{U} : \sigma(x) = D_i x\}$ for some $i$. Then $H$ is a subgroup of $\mathbb{G}_a$ and as suchit is not 1-based and has no definable subgroups if finite index.

Now we study prolongations and arc spaces of $H$.

First we will define algebraic arc spaces. For details see [5].

Let $K$ be a field, and $K^{(m)}$ the $K$-algebra $K[\epsilon]/(\epsilon^{m+1})$. If we identifying $K^{(m)}$ with $K \cdot 1 \oplus K \cdot \epsilon \ldots \oplus K \cdot \epsilon^m$, we see that the $K$-algebra $K^{(m)}$ is quantifier-free interpretable in $K$, if we encode elements of $K^{(m)}$ by $(m+1)$-tuples of $K$.

Let $V \subset \mathbb{A}^\ell$ be an algebraic variety defined over $K$. For $m \in \mathbb{N}$, we consider the set $V(K^{(m)})$ of $K^{(m)}$-points of $V$.

We may identify $V(K^{(m)})$ with a subvariety $\mathcal{A}_m V(K)$ of $\mathbb{A}^{(m+1)\ell}(K)$. The algebraic variety $\mathcal{A}_m V$ is the $m$-th arc bundle of $V$.

When $m = 1$ we can identify $\mathcal{A}_1 V$ with the tangent bundle $T(V)$.

If $r > m$, the natural projection $K^{(r)} \to K^{(m)}$ induces a map $V(K^{(r)}) \to V(K^{(m)})$, which induces a morphism $\rho_{r,m} : \mathcal{A}_r V \to \mathcal{A}_m V$.

Given a morphism of algebraic varieties $f : U \to V$ over $K$, the natural morphism $U(K^{(m)}) \to V(K^{(m)})$ induced by $f$ determines a morphism $\mathcal{A}_m f : \mathcal{A}_m U \to \mathcal{A}_m V$.

Let us write $\rho_m$ for $\rho_{m,0}$. For $a \in V(K)$ the $m$-th arc space of $V$ at $a$, $\mathcal{A}_m V_a$ is the fiber of $\rho_m$ over $a$.

Generic maps between algebraic varieties define epimorphisms between arc spaces:

**Lemma 3.1.** [5]. *Let $U, V$ be algebraic varieties defined over $K$, and let $f : U \to V$ be a generic map $K$. Let $a \in U(K)$ be non-singular such that $f(a)$ is non-singular and the rank of $df_a$ equals $\dim V$. Then for every $m \in \mathbb{N}$ the map $\mathcal{A}_m(f) : \mathcal{A}_m U_a(K) \to \mathcal{A}_m V_{f(a)}(K)$ is surjective.*

Arc spaces over non-singular points characterise algebraic varieties.

**Lemma 3.2.** [5]. *Let $U, V, W$ be algebraic varieties defined over $K$ such that $U, V \subset W$. Let $a \in U(K) \cap V(K)$ be non-singular. Then $U = V$ if and only if $\mathcal{A}_m U_a(K) = \mathcal{A}_m V_a(K)$ for all $m \in \mathbb{N}$.*

Now in order to define arc spaces for a $(\sigma, \Delta)$-variety we must define the differential prolongation of an algebraic variety. We use in this case the approach of [5] (section 2).

Let $(K, \Delta$ be a differential field. Define $K_m = K[\eta_1, \cdots, \eta_n]/(\eta_1, \cdots, \eta_n)^{m+1}$.

Let $E : K \to K_m$ defined by:

$$a \mapsto \sum_{0 \leq \alpha_1 + \cdots + \alpha_n} \frac{1}{\alpha_1! \cdots \alpha_n!} D_1^{\alpha_1} \cdots D_n^{\alpha_n}(a) \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}$$

If $V$ is an algebraic variety defined over $K$, the $m$-th $\Delta$-prolongation $\tau_m V$ of $V$ is the Weil restriction of $V \otimes_E K_m$ from $Spec(K_m)$ to $Spec(K)$.

The reduction maps $K_l \to K_m$ for $l \geq m$ imply that the prolongations form a projective system $\pi_{l,m} : \tau_l \to \tau_m$. If we identify $\tau_0$ with the identity and denote $\pi_m, 0$ as $\pi_m$, we obtain the projection $\pi_m : \tau_m V \to V$. The map $\nabla_m : V \to \tau_m V$ defined by

$$x \mapsto \sum_{0 \leq \alpha_1 + \cdots + \alpha_n} \frac{1}{\alpha_1! \cdots \alpha_n!} D_1^{\alpha_1} \cdots D_n^{\alpha_n}(x) \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}$$

is a section of $\pi_m$ and $\nabla_m(V)$ is $\Delta$-dense in $\tau_m V$.

$S_m(V)$ will denote the Zariski closure of $q_m(V)$ and $p_{l,m} : S_l(V) \to S_m(V)$ the natural projections for $l \geq m$.

We define $\Phi(V) = \tau_m(S_m(V))(= S_m(\tau_m V))$ and $\varphi = \nabla_m \circ q_m$.

We define now arc spaces of $(\sigma, \Delta)$-varieties. Extend $\sigma$ and $D_i$ to $K^{(m)}$ by defining $\sigma(\eta_j) = \eta_j$ and $D_i \eta_j = 0$. Since $\sigma$ and $D_i$ commute, we can identify $\mathcal{A}_r(S_m(V))(K)$ with $S_m(\mathcal{A}_r(V))(K)$ and we can assume that $\mathcal{A}_r(\Phi_m(V))(K) = \Phi_m(\mathcal{A}_m(V))(K)$.

If $X$ is a $(\sigma, \Delta)$-variety given as a $(\sigma, \Delta)$-closed subset of an algebraic variety $\bar{X}$, we define $\Phi_m(X)$ as the Zariski closure of $\psi_m(X)$ in $\Phi_m(\bar{X})$. $X$ is determined by the prolongation sequence $\{t_{l,m} : \Phi_l(X) \to \Phi_m(X)) : l \geq m\}$.

**Proposition 3.3.** *Let $(K, \sigma, \Delta)$ be a model of* $\mathrm{DCF}_m\mathrm{A}$. *Let $V$ be a $(\sigma, \Delta)$-variety given as a closed subvariety of an algebraic variety $\bar{V}$. Let $m \in \mathbb{N}$ and $a \in V(K)$ a non-singular point. Then $\{\mathcal{A}_m(t_{r,s}) : \mathcal{A}_m\Phi_r(V)_{\psi_r(a)} \to \mathcal{A}_m\Phi_s(V)_{\psi_s(a)}, r \geq s\}$ form the $(\sigma, \Delta)$-prolongation sequence of a $(\sigma, \Delta)$-subvariety of $\mathcal{A}_m\bar{V}_a$. We define the $m$-th arc space of $V$ at $a$, $\mathcal{A}_m V_a$, to be this subvariety. We have also that $\Phi_r(\mathcal{A}_m V_a) = \mathcal{A}_m\Phi_r(V)_{\psi_r(a)}$ for all $r$.*

Let $V$ be a $(\sigma, D)$-subvariety of the algebraic variety $\bar{V}$. We say that a point $a \in V$ is non-singular if, for all $l$, $\psi_l(a)$ is a non-singular point of $\Phi_l(V)$.

$(\sigma, \Delta)$-varieties are characterised by arc spaces over a non-singular point.

**Lemma 3.4.** *Let $U, V$ be two $(\sigma, D)$-subvarieties of an algebraic variety $\bar{V}$. Let $a \in U(K) \cap V(K)$ be a non-singular point of $U$. Then $U = V$ if and only if $\mathcal{A}_m\Phi_l(U)_{\psi_l(a)} = \mathcal{A}_m\Phi_l(V)_{\psi_l(a)}$ for all $m, l$.*

We can define a version of tanget space for $(\sigma, \Delta)$-varieties:

**Definition 3.5.** *Let $V$ be a $(\sigma, \Delta)$-variety and $a$ a non-singular point of $V$. We define the $(\sigma, \Delta)$-tangent space $T_{\sigma,\Delta}(V)_a$ of $V$ at $a$ as follows:*

*Let $P_r$ be a finite tuple of polynomials generating $I(\Phi_r(V)_{\psi_r(a)})$. Then $T_{\sigma,D}(V)_a$ is defined by the equations $J_{P_r}(\psi_r(a)) \cdot (\psi_r(Y)) = 0$.*

**Remark 3.6.** *Let $a$ be a non-singular point of the $(\sigma, \Delta)$-variety $V$. Then $T_{\sigma,\Delta}(V)_a$ is a subgroup of $\mathbb{G}_a^n(K)$.*

**Lemma 3.7.** *Let $V$ be a $(\sigma, \Delta)$-variety in $\mathbb{A}^l$ and $a$ a non-singular point of $V$. Then $\mathcal{A}_1 V_a$ is isomorphic to $T(V)_a$. Let $\bar{V}$ be the Zariski closure of $V(\mathcal{U})$ in $\mathbb{A}^l$ and $m \in \mathbb{N}$; then the map that identifies the fibers of $\mathcal{A}_{m+1}\bar{V}_a \to \mathcal{A}_m\bar{V}_a$ with $T(\bar{V})_a$ restricts to an isomorphism of the fibers of $\mathcal{A}_{m+1}V_a \to \mathcal{A}_m V_a$ with $T(V)_a$.*

Now back to $H$. It is easy to show that $SU(H) = \omega$, so its generic type is regular. As it is non 1-based, by 6.1.18 of [6] there are $K = acl(K) \subseteq \mathcal{U}$, $g_1, \cdots, g_l$ independent generic elements of $H$ over $K$, and $b = Cb(qftp(g_1, \cdots, g_l/Kb))$. Let $V$

be the locus of $(g_1, \cdots, g_l)$ over $K$. Then $V \subseteq H^l = T(H^l)_{(g_1,\cdots,g_l)}$ and $(g_1, \cdots, g_l)$ is non-singular point of $V$. Then $H' = T(V)_{(g_1,\cdots,g_l)}$ is a subgroup of $H^l$. We have also that $H' = \mathcal{A}_1(V)_{(g_1,\cdots,g_l)}$ and in general $(A)_m(V)_{(g,\cdots,g_l)} \subseteq \mathcal{A}_m(H^l)_{(g_1,\cdots,g_l)}$.

By 3.4 there is $k$ and there are $a_i \in \mathcal{A}_i(V)_{g_1,\cdots,g_l}$, $i = 1, \cdots, k$ such that $b$ is interalgebraic with $(a_1, \cdots, a_k)$ over $K(a_1, \cdots, a_l)$.

## References


[1] R. Bustamante Medina. Rank and dimension in difference-differential fields. *Notre Dame Journal of Formal Logic*, 52(4):403–414, 2011.

[2] R. Bustamante Medina. Zilber's dichotomy for differentially closed fields with an automorphism. *submitted*, 2020.

[3] Omar León Sánchez. On the model companion of partial differential fields with an automorphism. *Israel Journal of Mathematics*, 212(1):419–442, 5 2016.

[4] Tracey McGrail. The model theory of differential fields with finitely many commuting derivations. *J. Symbolic Logic*, 65(2):885–913, 06 2000.

[5] R. Moosa, Pillay A., and T. Scanlon. Differential arcs and regular types in differential fields. *J. Reine angew. Math.*, 2011(620):35–54, 2008.

[6] Frank O. Wagner. *Simple theories*. Kluwer Academic Publishers, Dordrecht, 2000.



Centro de Investigaciones en Matemática Pura y Aplicada, Escuela de Matemáticas, Universidad de Costa Rica, Sede Rodrigo Facio, 2060, San José, Costa Rica

*Email address*: ronald.bustamante@ucr.ac.cr